Probability Formulas and Concepts: WIP, not complete

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2024-03-20

Counting:

$$\text{Order matters, no repetition} \quad \frac{n!}{(n-r)!}$$
$$= nPr \text{Order matters, repetition} \quad n^r$$
$$\text{Order doesn't matter, no repetition} \quad \frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r}$$
$$= \frac{nPr}{r!}$$
$$\text{Order doesn't matter, repetition} \quad \frac{(n+r-1)!}{r!(n-1)!} =$$
$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

= number of ways to choose r objects from n objects

= number of ways to seat n people around a circle table = (n-1)!

$$\text{Probability of an event not occurring} \qquad P(\bar{E}) = 1 - P(E)$$

$$\text{Prob of event E} \qquad P(E) = \frac{\text{number of outcomes in E}}{\text{number of outcomes in }\Omega}$$

$$\text{PIE} \quad : \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{PIE 2} \quad : \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\text{Conditional} \quad : \quad P(A \cap \bar{B}) = P(A) - P(A \cap B) = P(A - B)$$

$$\text{A given B occurred} \quad : \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\text{Total Probability} \quad : \quad P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A|B_i)$$

$$\text{Bayes} \quad : \quad P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\text{Bayes with LTP} \quad : \quad P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}$$

$$\text{sample bayes for 2} \quad : \quad P(B) = P(A_i) \cdot P(B|A_i) + P(A_j) \cdot P(B|A_j)$$

Probability and Sample Space:

The sample space is denoted by Ω. When order matters, such as in cases where selection is done with repetition, the sample space is considered a uniform sample space.

If using "at least or at most" in a question, either use the complement or separate into cases of disjoint events (to avoid double counting)

Probability space:

probability function maps events $(subset of \Omega)$ to interval $[0, 1]$

satisfies two axioms $P(\Omega) = 1$ and if A,B are disjoint events:$(A \cap B = \emptyset)$

Properties of probability space:

$$P(\bar{A}) = 1 - P(A) \qquad \text{Probability of an event not occurring}$$
$$\bar{A} \cap A = \emptyset \qquad \text{A and not A are disjoint events}$$
$$P(A \cup \bar{A}) = P(\Omega) \qquad \text{A and not A are disjoint events}$$
$$P(A) + P(\bar{A}) = P(\Omega)$$
$$P(\emptyset) = 0 \qquad \emptyset \cup \bar{\emptyset} = \emptyset \cup U \text{U is universal set}$$
$$\emptyset = \bar{\Omega} \qquad \text{empty set is the complement of the universal set}$$
$$A \subseteq B \rightarrow P(A) \leq P(B)$$
$$A \subseteq B \rightarrow P(B) - P(A) = P(B - A) = P(B \cap \bar{A})$$

Overlapping events:

$$\text{for any events A,B}$$
$$P(A \cap B) \leq P(A) \leq P(A \cup B)$$
$$A \cap B \subseteq A \subseteq A \cup B \quad \text{and } A \cap B \subseteq B \subseteq A \cup B$$
$$\text{probability of unions:} \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\bar{A} \cap \bar{B} = \overline{A \cup B} \quad P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = \overline{P(A \cup B)}$$

Disjoint events:

$$P(A \cap B) = 0 \quad (A \cap B) = \emptyset$$

almost always dependent as we know that if A occurs, B doesn't occur

Independent events:

$$\text{Probability of two independent events} \quad P(A \cap B) = P(A) \cdot P(B)$$
$$P(A|B) = P(A) \qquad P(B|A) = P(B)$$
$$P(A|B) = P(A) \to P(B|A) = P(B)$$
$$\text{A,B independent} \leftrightarrow P(A \cap B) = P(A) \cdot P(B)$$
$$\text{for A, } \bar{B} \text{ independent}$$
$$A \cap B \cup A \cap \bar{B} = A$$

Law of total probability:

Let A1, A2, ..., An be disjoint events (a partition of the sample space)

Let B be any event

$$\to P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$
$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

Probability Functions:

Properties of expected values:

$$\text{for any constant } c: E[cx] = c \cdot E[x], E[x+c] = E[x] + c$$
$$\text{for any random variables } x, y: E[x+y] = E[x] + E[y]$$
$$\text{Linearity of Expectationfor any constants } a, b, c: \ E[ax+by+c] = a \cdot E[x] + b \cdot E[y] + c$$
$$\text{for independent random variables } v, z: E[vz] = E[v] \cdot E[z]$$

$$\text{E}[x] = \text{expected value of } x = \sum_k k \cdot P(X = k) \ \text{, game is fair if } E[x] = 0$$

over a long period of time playing game, no party should lose money, if $E[x] > 0$,
the player will be expected to profit on average, vs if $E[x] < 0$,
house expected to profit on average

$$\text{V}(x) = \text{variance of} X = E[x^2] - \mu^2$$
$$\text{E}[x] = \mu$$
$$\sigma(x) = \text{standard deviation of x} = \sqrt{\text{V}(x)}$$
$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) \text{for any r.v. X and Y}$$

Discrete Random Variable Distributions:

generic type of probability function which depends on one or more parameters

$$X \sim U(a,b) = \text{Uniform}$$
$a =$ minimum value $b =$ maximum value

$$X \sim B(n,p) = \text{Binomial}$$
$n =$ total number of trials $p =$ probability of success

With replacement, run same experiment each time, group is usually a population (huge number)

$$X \sim G(p) = \text{Geometric}$$
$p =$ probability of success $x =$ number of times experiment is done

with replacement, run same experiment each time till single success (and include in result)

$$X \sim P(\lambda) = \text{Poisson}$$
$\lambda =$ average number of successes per interval (np)

k is all possible value of x

if E[x] is the number of customers entering a store at a given time. K = prob * period of time

$$k = 0 : P(X = 0) = e^{-\lambda}$$

$$k = 1 : P(X = 1) = e^{-\lambda} \cdot \lambda$$

$$k = 2 : P(X = 2) = \frac{e^{-\lambda} \cdot \lambda^2}{2}$$

$X \sim H(N, n, p) =$ Hypergeometric

$p = M/N$ $N =$ total number of objects

$M =$ total number of special objects

$n =$ number of objects selected sample size

$p =$ probability of success

without replacement, run different experiment each time, group is fixed size

$X \sim NB(p, r) =$ Negative Binomial

$p =$ probability of success

$r =$ number of successes till stop, include final number in result

essentially its running geometric distribution r times

Approximations:

$$X \sim B(n, p) \approx \mathcal{N}(np, np(1-p))$$

$$P(X = k) \rightarrow k - 0.5 < X < k + 0.5$$

Misc:

Sum of infinite geometric sequence $\frac{a}{1-r}$

$a =$ first term

$r =$ common ratio

Sum of finite geometric sequence $\frac{a(1-r^n)}{1-r}$

$r \neq 1$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

$$1 - (1 - x) = x$$